数理解析研究所講究録
第 1545 巻 2007 年 1-12

Maximum principle via the iterated comparison function method

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1 Introduction

In this note, we present several maximum principles for L^p-viscosity solutions of fully nonlinear but uniformly elliptic/parabolic partial differential equations (PDEs for short). Our maximum principles are extentions of Aleksandrov-Bakelman-Pucci (ABP for short) type for elliptic case, and of ABP-Krylov-Tso for parabolic case.

We will work in a bounded open set $\Omega \subset \mathbf{R}^n$ for the elliptic case, and in $Q := \Omega \times (0, T]$ with a fixed $T > 0$ for the parabolic case. We will denote by B_r the open ball with center at the origin and the radus $r > 0$.

We denote by S^n the set of $n \times n$ symmetric matrices with the standard ordering $\leq$;

$$X \leq Y \iff \langle X\xi, \xi\rangle \leq 0 \quad \text{for } \forall \xi \in \mathbf{R}^n.$$

Throughout this paper, we at least suppose

$$p > \frac{n}{2} \text{ for the elliptic case and, } p > \frac{n+2}{2} \text{ for the parabolic case.}$$

We use the standard L^p-norm in a domain $U \subset \mathbf{R}^m$ ($m = n$ or $n+1$); $\|\cdot\|_{L^p(U)}$. However, we denote by $\|\cdot\|_p$ both $\|\cdot\|_{L^p(\Omega)}$ and $\|\cdot\|_{L^p(Q)}$ if there is no confusion. We also use the following notation:

$$L^p_+(U) = \{u \in L^p(U) \mid u \geq 0 \ a.e. \text{ in } U\}.$$

In what follows, given a function $f : U \to \mathbf{R}$, when we discuss it in a larger set V, we utilize the zero extention of f by the same f.

Freezing the uniform ellipticity constants $0 < \lambda \leq \Lambda$, we denote by $S^n_{\lambda,\Lambda}$ the set of all $A \in S^n$ such that $\lambda I \leq A \leq \Lambda I$.

Then, we define the Pucci operators $\mathcal{P}^\pm$: for $X \in S^n$,

$$\mathcal{P}^+(X) = \max\{-\mathrm{trace}(AX) \mid A \in S^n_{\lambda,\Lambda}\}, \quad \mathcal{P}^-(X) = \min\{-\mathrm{trace}(AX) \mid A \in S^n_{\lambda,\Lambda}\}.$$

An easy observation is that for $X, Y \in S^n$,

$$\mathcal{P}^-(X) + \mathcal{P}^-(Y) \leq \mathcal{P}^-(X+Y) \leq \mathcal{P}^-(X) + \mathcal{P}^+(Y) \leq \mathcal{P}^+(X+Y) \leq \mathcal{P}^+(X) + \mathcal{P}^+(Y),$$

which has a roll of "linearity" of fully nonlinear operators $\mathcal{P}^\pm$.

2 Elliptic case

Without loss of generality, we may suppose that $\Omega \subset B_1$.

Let us consider the most general PDEs of second-order in the elliptic case:

$$F(x, u, Du, D^2u) = f(x) \quad \text{in } \Omega, \tag{1}$$

where $F : \Omega \times \mathbf{R} \times \mathbf{R}^n \times S^n \to \mathbf{R}$ and $f : \Omega \to \mathbf{R}$ are given measurable functions, and F is continuous in the last three variables.

Definition. We call $u \in C(\Omega)$ an L^p-viscosity subsolution (resp., supersolution) of (1) if

$$ess \liminf_{y\to x}\{F(y, u(y), D\phi(y), D^2\phi(y)) - f(y)\} \le 0$$

$$\left(\text{resp.,} \quad ess \limsup_{y\to x}\{F(y, u(y), D\phi(y), D^2\phi(y)) - f(y)\} \ge 0\right)$$

whenever $\phi \in W^{2,p}_{\text{loc}}(\Omega)$ and $x \in \Omega$ is a local maximum (resp., minimum) point of $u - \phi$.

We then call $u \in C(\Omega)$ an L^p-viscosity solution of (1) if it is an L^p-viscosity subsolution and an L^p-viscosity supersolution of (1).

In order to memorize the right inequality, we will often say that u is an L^p-viscosity subsolution of

$$F(x, u, Du, D^2u) \le f(x) \quad etc.$$

Definition. We also call $u \in W^{2,p}_{\text{loc}}(\Omega)$ an L^p-strong subsolution (resp., supersolution) of (1) if u satisfies

$$F(x, u(x), Du(x), D^2u(x)) - f(x) \le 0 \quad (\text{resp.,} \ \ge 0) \quad \text{a.e. in } \Omega.$$

We then call $u \in W^{2,p}_{loc}(\Omega)$ an L^p-strong solution of (1) if the equality holds in the above.

Remark. Notice that we do not assume that $f \in L^p(\Omega)$. Thus, if u is an L^p-viscosity subsolution of (1), then it is also an L^q-viscosity subsolution of (1) provided $q \ge p$.

Now we suppose the uniform ellipticity for F:

$$\mathcal{P}^-(X - Y) \le F(x, r, p, X) - F(x, r, p, Y) \le \mathcal{P}^+(X - Y)$$

for $x \in \Omega$, $r \in \mathbf{R}$, $p \in \mathbf{R}^n$, and $X, Y \in S^n$. Typical examples of F are

$$F(x, r, p, X) = \max_{1\le i\le M} \min_{1\le j\le N}\{-\text{trace}(A(x; i, j)X) + \langle b(x; i, j), p\rangle + c(x; i, j)r\},$$

where for $M, N > 1$, functions $x \in \Omega \to A(x; i, j) \in S^n_{\lambda,\Lambda}$, $x \in \Omega \to b(x; i, j) \in \mathbf{R}^n$ and $x \to c(x; i, j)$ are measurable ($1 \le i \le M$, $1 \le j \le N$). Notice that the above F is non-convex and non-concave in general.

Under the uniform ellipticity assumption, we notice that if u is an L^p-viscosity subsolution of (1), then it is also an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2u) + F(x,u,Du,O) \leq f(x).$$

Therefore, for the sake of simplicity, instead of (1), we shall study the maximum principle for

$$\mathcal{P}^-(D^2u) - \mu(x)|Du| = f(x) \quad \text{in } \Omega. \tag{2}$$

Proposition 1. There exist $C_k = C_k(n,\lambda,\Lambda) > 0$ ($k = 1,2$) such that if $f, \mu \in L^n_+(\Omega)$, and $u \in C(\overline{\Omega}) \cap W^{2,n}_{\text{loc}}(\Omega)$ is an L^n-strong subsolution of (2), then we have

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u + C_1 \exp(C_2\|\mu\|_n)\|f\|_n. \tag{3}$$

Remark. In the above statement, we can replace $\|f\|_n$ by $\|f\|_{L^n(\Gamma[u])}$, where $\Gamma[u]$ is the upper contact set of u in Ω. See Gilbarg-Trudinger's book for the definition of $\Gamma[u]$.

From Proposition 1, it is trivial to obtain the corresponding result for L^p-strong supersolutions of

$$\mathcal{P}^+(D^2u) + \mu(x)|Du| \geq f(x) \quad \text{in } \Omega$$

by taking $v = -u$, which is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2v) - \mu(x)|Dv| \leq -f(x) \quad \text{in } \Omega.$$

Thus, we will give results only for subsolutions.

To utilize the "iterated comparison function method", we often use the following existence result for extremal equations (see [3]).

Proposition 2. There exists $p_0 = p_0(n,\Lambda/\lambda) \in [n/2,n)$ satisfying the following: If $p > p_0$ and Ω satisfy the uniform exterior cone condition, then there are $C = C(n,p,\lambda,\Lambda) > 0$ such that for $f \in L^p(\Omega)$, there is an L^p-strong solution $v \in C(\overline{\Omega}) \cap W^{2,p}_{\text{loc}}(\Omega)$ of

$$\begin{cases} \mathcal{P}^+(D^2v) = f(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases}$$

such that

$$-C\|f^-\|_p \leq v \leq C\|f^+\|_p \quad \text{in } \Omega.$$

Moreover, for each open set $\Omega' \Subset \Omega$, there is $C' = C'(n,p,\lambda,\Lambda,\text{dist}(\Omega',\partial\Omega)) > 0$ such that

$$\|v\|_{W^{2,p}(\Omega')} \leq C'\|f\|_p.$$

In this section, $A \Subset B$ means $\overline{A} \subset B$.

To show Proposition 1 for L^p-viscosity solutions, when μ is unbounded (*i.e.* $\mu \in L^q(\Omega)$ with $1 \leq q < \infty$ in our case), it is not trivial even if we suppose $f \equiv 0$. (When $\mu \in L^\infty(\Omega)$, we may apply a techinque as in our first paper [10].)

The next proposition is a restatement of Lemma 2.11 of [8] although our assumption that $\mathrm{supp}\mu \subset\subset \Omega$ seems restrictive (cf. [8]).

Proposition 3. Let Ω satisfy the uniform exterior cone condition. For

$$q \geq p > n \quad \text{or} \quad q > p = n, \tag{4}$$

we suppose $f \in L^p(\Omega)$, and $\mu \in L^q_+(\Omega)$ with $\mathrm{supp}\mu \subset\subset \Omega$. Then, there exist an L^p-strong supersolution u (resp., L^p-strong subsolution v) $\in C(\overline{\Omega}) \cap W^{2,p}_{\mathrm{loc}}(\Omega)$ of

$$\begin{cases} \mathcal{P}^-(D^2u) - \mu(x)|Du| \geq f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \left(\text{resp., } \begin{cases} \mathcal{P}^+(D^2v) + \mu(x)|Dv| \leq f(x) & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega \end{cases}\right)$$

such that

$$\|u\|_\infty \ (\text{resp., } \|v\|_\infty) \leq C_1 \exp(C_2\|\mu\|_n)\|f\|_n,$$

where C_1 and C_2 are the constants from Proposition 1. Moreover, for each open $\Omega' \subset\subset \Omega$, we have

$$\|u\|_{W^{2,p}(\Omega')} \left(\text{resp., } \|v\|_{W^{2,p}(\Omega')}\right) \leq C(n,p,\lambda,\Lambda,\|\mu\|_q, \mathrm{dist}\,(\Omega',\partial\Omega))\|f\|_p.$$

Now, we present an L^p-viscosity version of Proposition 1.

Proposition 4. Assume (4). Then, there exist $C_k = C_k(n,\lambda,\Lambda) > 0$ $(k = 1,2)$ such that if $f \in L^p_+(\Omega)$, $\mu \in L^q_+(\Omega)$, and $u \in C(\overline{\Omega})$ is an L^p-viscosity subsolution of (2), then we have

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u + C_1 \exp(C_2\|\mu\|_n)\|f\|_n.$$

Proof. Fix $\varepsilon > 0$. Recalling $\Omega \subset B_1$, from Proposition 2, we find an L^p-strong subsolution $v \in C(\overline{B}_2) \cap W^{2,p}_{loc}(B_2)$ of

$$\begin{cases} \mathcal{P}^+(D^2v) + \mu(x)|Dv| \leq -f(x) - \varepsilon & \text{in } B_2, \\ v = 0 & \text{on } \partial B_2 \end{cases}$$

such that

$$0 \leq -v \leq C_1 \exp(C_2\|\mu\|_n)(\|f\|_n + \varepsilon) \quad \text{in } B_2.$$

It is easy to check that $w := u + v$ is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2w) - \mu(x)|Dw| \leq -\varepsilon \quad \text{in } \Omega.$$

Hence, if w attains its maximum at $x \in \Omega$, the definition of L^p-viscosity subsolutions yields a contradiction. Thus, we have

$$\max_{\overline{\Omega}} w = \max_{\partial\Omega} w,$$

which implies that

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u + \max_{\Omega}(-v).$$

This gives the result follows by letting $\varepsilon \to 0$. □

Next, we consider the case of $p_0 < p < n$, which extends that in [8] and [9].

Theorem 5. Assume $p_0 < p < n < q$, and $m = 1$. There exist an integer $N = N(n,p,q)$ and $C = C(n,\lambda,\Lambda,p,q) > 0$ such that if $f \in L^p_+(\Omega)$, $\mu \in L^q_+(\Omega)$, and $u \in C(\overline{\Omega})$ is an L^p-viscosity subsolution of (2), then we have

$$\max_{\overline{\Omega}} u \le \max_{\partial\Omega} u + C\left\{\exp(C\|\mu\|_n)\|\mu\|_q^N + \sum_{k=0}^{N-1}\|\mu\|_q^k\right\}\|f\|_p.$$

Idea of proof. Due to Proposition 2, we find an L^p-strong solution $v_1 \in C(\overline{B}_{R_1}) \cap W^{2,p}_{loc}(B_{R_2})$ of

$$\begin{cases} \mathcal{P}^+(D^2v_1) = -f(x) & \text{in } B_2, \\ v_1 = 0 & \text{on } \partial B_2 \end{cases}$$

such that $0 \le -v_1 \le C\|f\|_p$ in B_2. By the Sobolev embedding, we have

$$\|Dv_1\|_{L^{p^*}(B_{3/2})} \le C\|f\|_p. \tag{5}$$

Here and later, for $n > p > 1$,

$$p^* = \frac{np}{n-p} > 0.$$

We will also use $C > 0$ to denote various universal constants.

By setting $w_1 = u + v_1$ in Ω, it is easy to see that w_1 is an L^p-viscosity subsolution of

$$\mathcal{P}^-(D^2w_1) - \mu(x)|Dw_1| \le \mu(x)|Dv_1(x)| =: f_2(x) \quad \text{in } \Omega.$$

By (5) and the Hölder inequality yield

$$\|f_2\|_{L^{q_1}(B_{3/2})} \le \|\mu\|_q\|Dv_1\|_{L^{p^*}(B_{3/2})} \le C\|\mu\|_q\|f\|_p,$$

where $q_1 = npq/\{(n-p)q + pn\}$. Note $q_1 > p$.

Let us suppose $q_1 > n$; $p > nq/(2q-n)$. In view of Proposition 4, we have

$$\max_{\overline{\Omega}} w_1 \le \max_{\partial\Omega} w_1 + C_1\exp(C_2\|\mu\|_n)\|f_2\|_{q_1},$$

which implies

$$\begin{aligned} \max_{\overline{\Omega}} u &\le \max_{\overline{\Omega}} w_1 + \max_{\overline{\Omega}}(-v_1) \\ &\le \max_{\partial\Omega} u + C\|f\|_p + C_1C\exp(C_2\|\mu\|_n)\|\mu\|_q\|f\|_p. \end{aligned}$$

If $q_1 \le n$, then we use the L^{q_1}-strong solution $v_2 \in C(\overline{B}_{3/2}) \cap W^{2,q_1}_{loc}(B_{3/2})$ of

$$\begin{cases} \mathcal{P}^+(D^2v_2) = -f_2(x) & \text{in } B_{3/2}, \\ v_2 = 0 & \text{on } \partial B_{3/2} \end{cases}$$

to derive the eqaution satisfied by $w_2 := w_1 + v_2$;

$$\mathcal{P}^-(D^2 w_2) - \mu(x)|Dw_2| \leq f_3(x),$$

where $f_3 \in L^{q_2}(B_{5/4})$ with $q_2 > q_1$. We keep on this procedure to arrive the situation $q_N > n$. Thus, we may apply Proposition 4 to conclude our result. □

Next, for $m > 1$, we consider the PDE

$$\mathcal{P}^-(D^2 u) - \mu(x)|Du|^m = f(x) \quad \text{in } \Omega. \tag{6}$$

In order to show the maximum principle for (6), we need some restrictions as in [10] because there is a counter-example (see [11]).

Theorem 6. Assume $n < p \leq q$, and $m > 1$. Then, there exist $\delta = \delta(n, \lambda, \Lambda, m, p, q) > 0$ and $C = C(n, \lambda, \Lambda, m, p, q) > 0$ such that if $f \in L^p_+(\Omega)$, $\mu \in L^q_+(\Omega)$,

$$\|f\|_p^{m-1} \|\mu\|_q \leq \delta,$$

and $u \in C(\overline{\Omega})$ is an L^p-viscosity subsolution of (6), then we have

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u + C\left(\|f\|_p + \|f\|_p^m \|\mu\|_q\right).$$

The idea of proof of Theorem 5 is a combination of those in [10] and Theorem 4.

Following the argument used in the proof of Theorem 5, we can now extend Theorem 6 to the case when $p \in (p_0, n]$.

Theorem 7. Assume $p_0 < p \leq n < q$, and $m > 1$. Denote $a_0 = 0$ and $a_k = 1 + m + \cdots + m^{k-1}$ for $k \geq 1$. Then, there exist an integer $N = N(n, m, p, q) \geq 1$, $\delta = \delta(n, \lambda, \Lambda, m, p, q) > 0$ and $C = C(n, \lambda, \Lambda, m, p, q) > 0$ such that if $f \in L^p_+(\Omega)$, $\mu \in L^q_+(\Omega)$,

$$p > \frac{nq(m-1)}{mq - n}, \tag{7}$$

$$\|f\|_p^{m^N(m-1)} \|\mu\|_q^{a_N(m-1)+1} \leq \delta,$$

and $u \in C(\overline{\Omega})$ is an L^p-viscosity subsolution of (6), then we have

$$\max_{\overline{\Omega}} u \leq \max_{\partial\Omega} u + C \sum_{k=0}^{N+1} \|\mu\|_q^{a_k} \|f\|_p^{m^k}.$$

Remark. When $1 < m \leq 2 - n/q$, (7) is automatically satisfied.

DIAGRAM 1 $\quad \mathcal{P}^-(D^2u) - \mu(x)|Du|^m \leq f(x) \Longrightarrow \max_{\overline{\Omega}} u - \max_{\partial\Omega} u \leq C \times \text{RHS}$

m	$\mu \in L^q,\ f \in L^p$	restriction	RHS
$m = 1$	$n < p \leq q < \infty$ or $n = p < q < \infty$	Nothing	$\exp(C\|\mu\|_n)\|f\|_n$
$m = 1$	$p_0 < p < n < q < \infty$	Nothing	$\left\{exp(C\|\mu\|_n)\|\mu\|_q^N + \sum_{k=0}^{N-1} \|\mu\|_q^k\right\} \|f\|_p$
$m > 1$	$n < p \leq q < \infty$	$\|f\|_p^{m-1}\|\mu\|_q < \exists\delta$	$\|f\|_p + \|f\|_p^m\|\mu\|_q$
$m > 1$	$p_0 < p \leq n < q < \infty$	$p > \dfrac{nq(m-1)}{mq-n}$, $\|f\|_p^{m^{\exists N}(m-1)}\|\mu\|_q^{a_N(m-1)+1} < \exists\delta$	$\sum_{k=0}^{N+1} \|\mu\|_q^{a_k}\|f\|_p^{m^k}$

Recall $a_k = 1 + m + \cdots m^{k-1}$.

We notice that when $m \geq 1$, $p_0 < p$ and $q = \infty$, we obtained the maximum principle with/without restriction in [10].

3 Parabolic equations

In this section, we consider parabolic PDEs in $Q := \Omega \times (0, T]$, where $\Omega \subset B_1$ again, and $0 < T \leq 1$ for simplicity. For $1 \leq p \leq \infty$, the parabolic Sobolev space $W^{2,1,p}(Q)$ is defined by

$$W^{2,1,p}(Q) = \left\{u \in L^p(Q) : u_t, Du, D^2u \in L^p(Q)\right\}.$$

In this section, we denote the parabolic boundary by $\partial_p Q := \Omega \times \{0\} \cup \partial\Omega \times [0, T]$.

We will also use the space $W^{2,1,p}_{\mathrm{loc}}(Q) = \{u : Q \to \mathbf{R} \ :\ u \in W^{2,1,p}(Q') \text{ for all } Q' \Subset Q\}$, where in this section, $Q' \Subset Q$ means $\mathrm{dist}(Q', \partial_p Q) > 0$.

The parabolic distance between (x, t) and (y, s) is defined by

$$\mathrm{dist}((x,t),(y,s)) = (|x-y|^2 + |t-s|)^{\frac{1}{2}}.$$

We recall the definition of L^p-viscosity solution of general fully nonlinear parabolic PDEs.

Definition. We call $u \in C(Q)$ an L^p-viscosity subsolution (resp., supersolution) of

$$u_t + F(x, t, u, Du, D^2u) = f(x, t) \quad \text{in } Q, \tag{8}$$

if

$$ess \liminf_{(y,s)\in Q \to (x,t)} \left\{\phi_t(y,s) + F(y,s,u(y,s),D\phi(y,s),D^2\phi(y,s)) - f(y,s)\right\} \leq 0$$

$$\left(\text{resp.,} \quad ess \limsup_{(y,s)\in Q\to(x,t)} \left\{\phi_t(y,s) + F(y,s,u(y,s),D\phi(y,s),D^2\phi(y,s)) - f(y,s)\right\} \geq 0\right)$$

whenever $\phi \in W^{2,1,p}_{\mathrm{loc}}(Q)$ and $(x,t) \in \Omega \times (0,T)$ is a local maximum (resp., minimum) point of $u - \phi$.

We call $u \in C(Q)$ an L^p-viscosity solution of (8) if it is an L^p-viscosity sub- and super-solution of (8).

As in the elliptic case, we call $u \in W^{2,1,p}_{\mathrm{loc}}(Q)$ an L^p-strong solution of (8) if u satisfies

$$u_t(x,t) + F(x,t,u(x,t),Du(x,t),D^2u(x,t)) = f(x,t) \quad a.e. \text{ in } Q.$$

As in section 2, we will establish maximum principles for the following simpler parabolic PDE

$$u_t + \mathcal{P}^-(D^2u) - \mu(x,t)|Du|^m = f(x,t) \quad \text{in } Q, \tag{9}$$

where $m \geq 1$.

The following version of maximum principle can be derived from [13].

Proposition 8. Let $m = 1$, $f \in L^{n+1}_+(Q)$ and $\mu \in L^{n+1}_+(Q)$. Then, there exist $C_k = C_k(n,\lambda,\Lambda) > 0$ $(k = 1,2)$ such that if $u \in C(\overline{Q}) \cap W^{2,1,n+1}_{\mathrm{loc}}(Q)$ is an L^{n+1}-strong subsolution of (9), then we have

$$\max_{\overline{Q}} \leq \max_{\partial_p Q} u + C_1 \exp(C_2\|\mu\|_{n+1})\|f\|_{n+1}.$$

We may also refine the above estimate using the upper contact set (see [13] for the details).

In this section, we fix $p_1 = p_1(n,\Lambda/\lambda) \in ((n+2)/2, n+1)$ to be the "parabolic" constant that gives the range of exponents for which the following generalized maximum principle holds (see [7]): for $p > p_1$, there is a constant $C = C(n,\lambda,\Lambda,p)$ such that if $f \in L^p(Q)$ and $u \in C(\overline{Q}) \cap W^{2,1,p}_{\mathrm{loc}}(Q)$ satisfies $u_t + \mathcal{P}^-(D^2u) \leq f(x,t)$ *a.e.* in Q, then we have

$$\max_{\overline{Q}} u \leq \max_{\partial_p Q} u + C\|f^+\|_p.$$

We recall results on solvability of extremal equations and on estimates of Du.

Proposition 9. Let $p > p_1$. There exists $C = C(n,\lambda,\Lambda,p) > 0$ such that for $f \in L^p(Q)$, there exists an L^p-strong solution $u \in C(\overline{Q}) \cap W^{2,1,p}_{\mathrm{loc}}(Q)$ of

$$\begin{cases} u_t + \mathcal{P}^+(D^2u) = f(x,t) & \text{in } Q, \\ u = 0 & \text{on } \partial_p Q, \end{cases} \tag{10}$$

such that

$$-C\|f^-\|_p \leq u \leq C\|f^+\|_p \quad \text{in } Q.$$

Moreover, for each set $Q' \Subset Q$, there exists $C' = C'(n, \lambda, \Lambda, p, \mathrm{dist}(Q', \partial_p Q)) > 0$ such that

$$\|u\|_{W^{2,1,p}(Q')} \leq C' \|f\|_p.$$

To study (9), as in the elliptic case, it is important to know the L^∞-estimate of Du from the embeddings:

Proposition 10. (cf. Theorem 7.3 in [5]) Let $p > p_1$. For each set $Q' \Subset Q$, there exists $C = C(n, \lambda, \Lambda, p, \mathrm{dist}(Q', \partial_p Q)) > 0$ such that if $u \in C(\overline{Q}) \cap W^{2,1,p}_{\mathrm{loc}}(Q)$ is an L^p-strong solution of (9), then we have

$$\|Du\|_{L^\infty(Q')} \leq C(\|u\|_{L^\infty(\partial_p Q)} + \|f\|_p) \quad \text{if } p > n+2,$$

$$\|Du\|_{L^{p^*}(Q')} \leq C(\|u\|_{L^\infty(\partial_p Q)} + \|f\|_p) \quad \text{if } p \in (p_1, n+2).$$

Here and later, p^* above is defined by

$$p^* = \frac{p(n+2)}{n+2-p} \quad \text{for } p < n+2.$$

We present a parabolic version of Proposition 3:

Proposition 11. Let Ω satisfy the uniform exterior cone condition.

$$q \geq p > n+2 \quad \text{or} \quad q > p = n+2, \tag{11}$$

$f \in L^p_+(Q)$, and let $\psi \in C(\partial_p Q)$. Let $\mu \in L^q_+(Q)$ satisfy $\mathrm{supp}\mu \Subset Q$. Then, there exist L^p-strong subsolutions u (resp., L^p-strong supersolution v) $\in C(\overline{Q}) \cap W^{2,p}_{\mathrm{loc}}(Q)$ of

$$\begin{cases} u_t + \mathcal{P}^-(D^2 u) - \mu(x,t)|Du| \geq f(x,t) & \text{in } Q, \\ u = 0 & \text{on } \partial_p Q, \end{cases}$$

$$\left(\text{resp., } \begin{cases} v_t + \mathcal{P}^+(D^2 v) + \mu(x,t)|Dv| \leq f(x,t) & \text{in } Q, \\ v = 0 & \text{on } \partial_p Q \end{cases}\right)$$

such that

$$\|u\|_{L^\infty(Q)} \left(\text{resp., } \|v\|_{L^\infty(Q)}\right) \leq C_1 \exp(C_2 \|\mu\|_{n+1}) \|f\|_{n+1},$$

where C_1 and C_2 are constants from Proposition 8. For each $Q' \Subset Q$, we have

$$\|u\|_{W^{2,1,p}(Q')} \left(\text{resp., } \|v\|_{W^{2,1,p}(Q')}\right) \leq C(n, p, \lambda, \Lambda, \|\mu\|_{L^q(Q)}, \mathrm{dist}\,(Q', \partial_p Q)) \|f\|_{L^p(Q)}. \tag{12}$$

By following the proof of Proposition 4, Proposition 10 allows us to obtain the following maximum principle.

Proposition 12. Assume (11) and $m = 1$. Then, there exist $C_k = C_k(n, \lambda, \Lambda) > 0$ $(k = 1, 2)$ such that if $f \in L^p_+(Q)$, $\mu \in L^q_+(Q)$, and $u \in C(\overline{Q})$ is an L^p-viscosity subsolution of (9), then we have

$$\max_{\overline{Q}} \leq \max_{\partial_p Q} u + C_1 \exp(C_2 \|\mu\|_{n+1}) \|f\|_{n+1}.$$

We first show that if $\mu \in L^\infty_+(Q)$, then even for $m > 1$, we do not need to assume that $\|\mu\|_\infty$ or $\|f\|_p$ is small. Recall that such a restriction is necessary in the elliptic case as discussed in [10] and [11].

Theorem 13. Assume $n + 2 < p \leq q$, and $m \geq 1$. Then, there exixts $C = C(n, \lambda, \Lambda, p, m) > 0$ such that if $f \in L^p_+(Q)$, $\mu \in L^\infty_+(Q)$, and $u \in C(\overline{Q})$ is an L^p-viscosity subsolution of (9), then we have

$$\max_{\overline{Q}} u \leq \max_{\partial_p Q} u + C(\|f\|_p + \|\mu\|_\infty \|f\|_p^m).$$

We next extend Theorem 13 to the case $p \in (p_1, n+2]$.

Theorem 14. Assume $p_1 < p \leq n + 2 < q$, and $m \geq 1$. Then, there exist an integer $N = N(n, p, m) \geq 1$ and $C = C(n, \lambda, \Lambda, p, m) > 0$ such that if $f \in L^p_+(Q)$, $\mu \in L^\infty_+(Q)$,

$$p > \frac{(m-1)(n+2)}{m}, \tag{13}$$

and $u \in C(\overline{Q})$ is an L^p-viscosity subsolution of (9), then we have

$$\max_{\overline{Q}} u \leq \max_{\partial_p Q} u + C\left(\|f\|_p^m \sum_{k=0}^{N} \|\mu\|_p^k + \|\mu\|_\infty^{mN+1} \|f\|_p^{m^2}\right).$$

Remark. We remark that when $m \in [1, 2]$, since $p_1 \geq (n+2)/2 \geq (m-1)(n+2)/m$, the restriction (13) is not necessary.

Next, we discuss the case when $m = 1$ in (9) but $\mu \in L^q(Q)$ with $q > n + 2$.

Theorem 15. Assume $p_1 < p \leq n + 2 < q$, and $m = 1$. Then, there exist an integer $N = N(n, p, q) \geq 1$ and $C = C(n, \lambda, \Lambda, p, q) > 0$ such that if $f \in L^p_+(Q)$, $\mu \in L^q_+(Q)$, and $u \in C(\overline{Q})$ is an L^p-viscosity subsolution of (9), then we have

$$\max_{\overline{Q}} u \leq \max_{\partial_p Q} u + C\left\{\exp(C\|\mu\|_{n+1})\|\mu\|_q^N + \sum_{k=0}^{N-1} \|\mu\|_q^k\right\} \|f\|_p.$$

Finally, we give sufficient conditions under which the maximum principle for (9) with $m > 1$ holds true. The first result corresponds to Theorem 6 for elliptic PDEs.

Theorem 16. Assume $n+2 < p \leq q$, and $m > 1$. Then, there exist $\delta = \delta(n, \lambda, \Lambda, m, p, q) > 0$ and $C = C(n, \lambda, \Lambda, m, p, q) > 0$ such that if $f \in L^p_+(Q)$, $\mu \in L^q_+(Q)$,

$$\|f\|_p^{m-1} \|\mu\|_q < \delta,$$

and $u \in C(\overline{Q})$ is an L^p-viscosity subsolution of (9), then we have

$$\max_{\overline{Q}} u \leq \max_{\partial_p Q} u + C(\|f\|_p + \|\mu\|_q \|f\|_p^m).$$

Our last result extends Theorem 16 to the case of $p_1 < p \leq n+2$.

Theorem 17. Assume $p_1 < p \leq n+2 < q$. Denote $a_0 = 0$ and $a_k = 1 + m + \cdots + m^{k-1}$ for $k \geq 1$. Then, there exist an integer $N = N(n,m,p,q) \geq 1$, $\delta = \delta(n,\lambda,\Lambda,m,p,q) > 0$ and $C = C(n,\lambda,\Lambda,m,p,q) > 0$ such that if $f \in L^p_+(Q)$, $\mu \in L^q_+(Q)$,

$$p > \frac{(m-1)q(n+2)}{mq-n-2}, \tag{14}$$

and $u \in C(\overline{Q})$ is an L^p-viscosity subsolution of (9),

$$\|f\|_p^{m^N(m-1)} \|\mu\|_q^{a_N(m-1)+1} \leq \delta,$$

then we have

$$\max_{\overline{Q}} u \leq \max_{\partial_p Q} u + C \left\{ \sum_{k=0}^{N+1} \|\mu\|_q^{a_k} \|f\|_p^{m^k} \right\}.$$

Remark. If $1 < m < 2 - (n+2)/q$, the restriction (14) is not necessary.

DIAGRAM 2 $u_t + \mathcal{P}^-(D^2u) - \mu(x,t)|Du|^m \leq f(x,t) \Longrightarrow \max_{\overline{Q}} u - \max_{\partial_p Q} u \leq C \times \mathrm{RHS}$

m	$\mu \in L^q,\ f \in L^p$	restriction	RHS
$m \geq 1$	$n+2 < p, q = \infty$	Nothing	$\|f\|_p + \|\mu\|_\infty \|f\|_p^m$
$m \geq 1$	$p_1 < p \leq n+2, q = \infty$	$p > \dfrac{(m-1)(n+2)}{m}$	$\|f\|_p^m \sum_{k=0}^{\exists N} \|\mu\|_\infty^k + \|\mu\|_\infty^{mN+1} \|f\|_p^{m^2}$
$m = 1$	$n+2 < p \leq q < \infty$ or $n+2 = p < q < \infty$	Nothing	$\exp(C\|\mu\|_{n+1})\|f\|_{n+1}$
$m = 1$	$p_1 < p \leq n+2 < q < \infty$	Nothing	$\left\{\exp(C\|\mu\|_{n+1})\|\mu\|_q^{\exists N} + \sum_{k=0}^{N-1} \|\mu\|_q^k \right\} \|f\|_p$
$m > 1$	$n+2 < p \leq q < \infty$	$\|f\|_p^{m-1}\|\mu\|_q < \exists\delta$	$\|f\|_p + \|f\|_p^m \|\mu\|_q$
$m > 1$	$p_1 < p \leq n+2 < q < \infty$	$p > \dfrac{(m-1)q(n+2)}{mq-n-2}$, $\|f\|_p^{m^{\exists N}(m-1)} \|\mu\|_q^{a_N(m-1)+1} < \exists\delta$	$\sum_{k=0}^{N+1} \|\mu\|_q^{a_k} \|f\|_p^{m^k}$

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